CT-duality as a local property of the world-sheet *

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Abstract

In the present article, we study the local features of the world-sheet in the case when probe bosonic string moves in antisymmetric background field. We generalize the geometry of surfaces embedded in space-time to the case when the torsion is present. We define the mean extrinsic curvature for spaces with Minkowski signature and introduce the concept of mean torsion. Its orthogonal projection defines the dual mean extrinsic curvature. In this language, the field equation is just the equality of mean extrinsic curvature and extrinsic mean torsion, which we call CT-duality. To the world-sheet described by this relation we will refer as CT-dual surface.

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1 Introduction

In general relativity, the action for the point particle is proportional to the length between initial and final points. For the geodesic line, which is solution of the equation of motion, the action is minimal. On the other hand, there exists a definition of the geodesic in term of local properties of the trajectory in the neighborhood of every given point. This definition is equivalent to the previous one, because it produces the same equation of motion. According to local features, geodesic is self-parallel line, which means that tangent vector after parallel transport along this line remains tangent.

In the string theory the action is area of the world-sheet between given initial and final positions of the string. The solution of the equations of motion, by definition, is minimal surface. What is definition of the minimal surface in term of local properties of the surface in the neighborhood of every given point? The direct generalization of "self-parallel" surface is impossible, because even initial and final positions of the string are not

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necessary self-parallel. The local features of minimal surface is known in the literature and this is the condition that all mean extrinsic curvatures vanish [1, 2].

We are interested in more general case, when the string propagates in nontrivial massless background, which beside metric includes also antisymmetric tensor. The corresponding Lagrangian of the theory has been used in the literature [3]-[7], in order to derive the classical space-time equations of motion from the world-sheet quantum conformal invariance. The goal of the present paper is to describe the world-sheet equations of motion in terms of local features, in the presents of antisymmetric field.

In Sec. 2, we formulate the bosonic string theory which we are going to consider, and shortly repeat some results of ref. [8], which we will need later.

In Sec. 3, we consider general theory of surfaces embedded into Riemann-Cartan space-time. We define the induced world-sheet variables: metric tensor and connection, and extrinsic one: second fundamental form (SFF). The world-sheet tangent vector, after parallel transport along world-sheet line with space-time connection, is not necessarily a tangent vector. Its world-sheet projection defines the induced connection and its normal projection defines the SFF.

In Secs. 4 and 5. we explain the local properties of the world-sheet in the presence of background fields. First we consider the case in the absence of antisymmetric field. We precisely define mean extrinsic curvature (MEC) in Minkowski space-time, and support the result of refs.[1, 2].

We generalized the concept of the SFF and of the MEC to the case where the space-time has nontrivial torsion. We define the mean torsion, ${}^\circ T^\mu$, and show that its orthogonal projection, ${}^\circ T_i$, is dual mean extrinsic curvature (DMEC), ${}^\circ T_i = {}^* H_i$. This enables us to introduce the self-dual (self-antidual) condition, ${}^\circ H_i = \pm {}^* H_i$, to which we will refer as CT-duality. Finally, we prove the main result, that the field equations in the presence of antisymmetric field have the form of CT-duality and that according to the local properties the world-sheet is self-dual surface.

Appendix A is devoted to the world-sheet geometry.

2 Formulation of the theory

Let us consider the action [4]-[7]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.1}$$

which describes bosonic string propagation in x^μ-dependent background fields: metric $G_{\mu\nu}$ and antisymmetric tensor field $B_{\mu\nu} = -B_{\nu\mu}$. Let $x^\mu(\xi)\,(\mu = 0, 1, ..., D-1)$ be the coordinates of the D dimensional space-time M_D and $\xi^\alpha\,(\xi^0 = \tau, \xi^1 = \sigma)$ the coordinates

of two dimensional world-sheet Σ, spanned by the string. The corresponding derivatives we will denote as $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial_\alpha \equiv \frac{\partial}{\partial \xi^\alpha}$ and the intrinsic world-sheet metric by $g_{\alpha\beta}$.

We will briefly review some results of the ref. [8], adapted for the present case without dilaton field. It is useful to define the currents

$$j_{\pm\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu} x^{\nu\prime} \,, \qquad \Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu} \,, \tag{2.2}$$

where π_μ is canonical momentum of the coordinate x^μ.

The τ and σ derivatives of the coordinate x^μ can be expressed in terms of the corresponding currents

$$\dot{x}^\mu = \frac{G^{\mu\nu}}{2\kappa}(h^- j_{-\nu} - h^+ j_{+\nu}) \,, \qquad x^{\mu\prime} = \frac{G^{\mu\nu}}{2\kappa}(j_{+\nu} - j_{-\nu}) \,, \tag{2.3}$$

where the components of the intrinsic metric tensor, $h^\pm$, are define in the App. A.

The canonical Hamiltonian density, $\mathcal{H}_c = h^- T_- + h^+ T_+$, and the energy momentum tensor components

$$T_\pm = \mp \frac{1}{4\kappa} G^{\mu\nu} j_{\pm\mu} j_{\pm\nu} \,, \tag{2.4}$$

have the standard forms. The equations of motion, for the action (2.1) is

$$[x^\mu] \equiv \nabla_- \partial_+ x^\mu + \Gamma^\mu_{-\rho\sigma} \partial_+ x^\rho \partial_- x^\sigma = 0 \,, \tag{2.5}$$

$$[h^\pm] \equiv G_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu = 0 \,, \tag{2.6}$$

where the world-sheet covariant derivatives, $\nabla_\pm$, are defined in (A.6). The expression in the $[x^\mu]$ equation is of the form

$$\Gamma^\rho_{\pm\nu\mu} = \Gamma^\rho_{\nu\mu} \pm B^\rho_{\nu\mu} \,, \tag{2.7}$$

where

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = D_\mu B_{\nu\rho} + D_\nu B_{\rho\mu} + D_\rho B_{\mu\nu} \,, \tag{2.8}$$

is the field strength of the antisymmetric tensor. It is a generalized connection, which full geometrical interpretation has been investigated in [3]. Under space-time general coordinate transformations the expression $\Gamma^\rho_{\pm\nu\mu}$ transforms as a connection.

As a consequence of the symmetry relations $\Gamma^\mu_{\mp\rho\sigma} = \Gamma^\mu_{\pm\sigma\rho}$, we can rewrite eq. (2.5) in the form $[x^\mu] \equiv \nabla_+ \partial_- x^\mu + \Gamma^\mu_{+\rho\sigma} \partial_- x^\rho \partial_+ x^\sigma = 0$. So, all considerations we can also apply to $\Gamma^\mu_{+\rho\sigma}$.

3 Geometry of surfaces embedded in Riemann-Cartan space-time

The geometry of surfaces, when the world-sheet is embedded in curved space-time, has been investigated in the literature, see [1, 2]. In this section we will generalize these results for the space-times with nontrivial torsion.

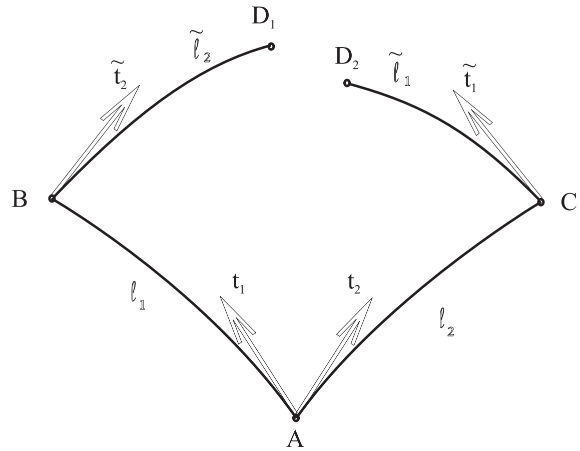

Figure 1: *Geometrical meaning of the torsion.*

3.1 Riemann-Cartan geometry

Let us first fix notation and shortly repeat some definitions of refs. [3], [9] and [10].

The affine linear connection, ${}^\circ\Gamma^\mu_{\rho\sigma}$, defines the rule for parallel transport of the vector $V^\mu(x)$, from the point x to the point $x+dx$, as $V^\mu(x) \to {}^\circ V^\mu_{\|} = V^\mu + {}^\circ\delta V^\mu$, where

$$ {}^\circ\delta V^\mu = -{}^\circ\Gamma^\mu_{\rho\sigma} V^\rho dx^\sigma \,. \tag{3.1} $$

The covariant derivative is defined as

$$ {}^\circ D V^\mu = V^\mu(x+dx) - {}^\circ V^\mu_{\|} \equiv {}^\circ D_\nu V^\mu dx^\nu \,, \tag{3.2} $$

where ${}^\circ D_\nu V^\mu = \partial_\nu V^\mu + {}^\circ\Gamma^\mu_{\rho\nu} V^\rho$.

The connection is not necessary symmetric in the lower indices, and its antisymmetric part is the torsion

$$ {}^\circ T^\rho_{\mu\nu} = {}^\circ\Gamma^\rho_{\mu\nu} - {}^\circ\Gamma^\rho_{\nu\mu} \,. \tag{3.3} $$

It has a simple geometrical interpretation which we will need later. Let us perform parallel transport of the unit tangent vectors $t_1^\mu \, (t_2^\mu)$ along geodesics $\ell_2 \, (\ell_1)$ at the distances $d\ell_2 \, (d\ell_1)$, respectively. The final vectors we denote by $\tilde{t}_1^\mu \, (\tilde{t}_2^\mu)$. They define directions of the geodesics $\tilde{\ell}_1 \, (\tilde{\ell}_2)$, which ends at the points $D_2 \, (D_1)$, at the distances $d\ell_1 \, (d\ell_2)$, (Fig. 1). The difference of the coordinates at the points D_2 and D_1 is proportional to the torsion

$$ x^\mu(D_2) - x^\mu(D_1) = {}^\circ T^\mu{}_{\rho\sigma} \, t_1^\rho \, t_2^\sigma \, d\ell_1 \, d\ell_2 \,. \tag{3.4} $$

In fact *the torsion measures the non-closure of the "rectangle" ABCD*.

In the present case, the decomposition of the connection has a form

$$ {}^\circ\Gamma_{\mu,\rho\sigma} = \Gamma_{\mu,\rho\sigma} + {}^\circ K_{\mu\rho\sigma} \,. \tag{3.5} $$

The first term is the Christoffel connection, $\Gamma_{\mu,\rho\sigma} = \frac{1}{2}\partial_{\{\mu}G_{\rho\sigma\}}$, and the second one is the contortion, ${}^\circ K_{\mu\rho\sigma} = \frac{1}{2}{}^\circ T_{\{\sigma\mu\rho\}}$, where we introduce the Schouten braces according to the relation $\{\mu\rho\sigma\} = \sigma\mu\rho + \rho\sigma\mu - \mu\rho\sigma$.

The first term is symmetric in ρ, σ indices. In the second term we can separate symmetric and antisymmetric parts, ${}^\circ K_{\mu\rho\sigma} = {}^\circ K_{\mu(\rho\sigma)} + \frac{1}{2}{}^\circ T_{\mu\rho\sigma}$, where he symmetric part of the arbitrary tensor $X_{\mu\nu}$ we denote as $X_{(\mu\nu)} \equiv \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu})$. Consequently, we have

$$ {}^\circ\Gamma^\mu_{\rho\sigma} = {}^\circ\Gamma^\mu_{(\rho\sigma)} + \frac{1}{2}{}^\circ T^\mu_{\rho\sigma} \,. \tag{3.6} $$

The antisymmetric part of the connection (2.7) is the Rieman-Cartan torsion, seen by the string

$$ T\pm^\rho_{\mu\nu} = \Gamma\pm^\rho_{\mu\nu} - \Gamma\pm^\rho_{\nu\mu} = \pm 2B^\rho_{\mu\nu} \,. \tag{3.7} $$

It is proportional to the field strength of the antisymmetric tensor field $B_{\mu\nu}$. In this case the contortion is proportional to the torsion $K_{\pm\mu\nu\rho} = \frac{1}{2}T_{\pm\mu\nu\rho} = \pm B_{\mu\nu\rho}$. So, the expressions (3.5) and (3.6) turn to (2.7).

3.2 Induced and extrinsic geometry

We will use the local space-time basis, relating with the coordinate one by the vielbein $E^\mu_A = \{\partial_\alpha x^\mu, n^\mu_i\}$. Here, $\partial_\alpha x^\mu = \{\dot{x}^\mu, x'^\mu\}$ is local world-sheet basis and n^μ_i ($i = 2, 3, ..., D-1$) are local unit vectors, normal to the world-sheet.

The world-sheet **induced metric tensor**

$$ G_{\alpha\beta} = G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{3.8} $$

is defined by the requirement, that any world-sheet interval has the same length, measured by the target space metric or by the induced one.

Similarly, the induced metric of a $D-2$ dimensional space, normal to the world-sheet, is $G_{ij} = G_{\mu\nu}n^\mu_i n^\nu_j$. The mixed induced metric tensor $G_{\alpha i} = G_{\mu\nu}\partial_\alpha x^\mu n^\nu_i$, vanishes by definition.

The world-sheet projection and the orthogonal projection of the arbitrary space-time covector V_μ, we will denote as

$$ v_\alpha = \partial_\alpha x^\mu V_\mu \,, \qquad v_i = n^\mu_i V_\mu \,. \tag{3.9} $$

In the space-time basis, tangent and normal vectors to the world-sheet Σ can be expressed respectively as

$$ V^\mu_\Sigma = \partial_\alpha x^\mu v^\alpha \,, \qquad V^\mu_\perp = n^\mu_i v^i \,. \tag{3.10} $$

Let us perform the parallel transport of the covector V^Σ_μ along world-sheet line, from the point ξ^α to the point $\xi^\alpha + d\xi^\alpha$, using the space-time connection ${}^\circ\Gamma^\nu_{\mu\rho}$ (Fig. 2). We obtain the covector

$$ {}^\circ V^\Sigma_{\|\mu} = V^\Sigma_\mu + {}^\circ\delta V^\Sigma_\mu \,, \tag{3.11} $$

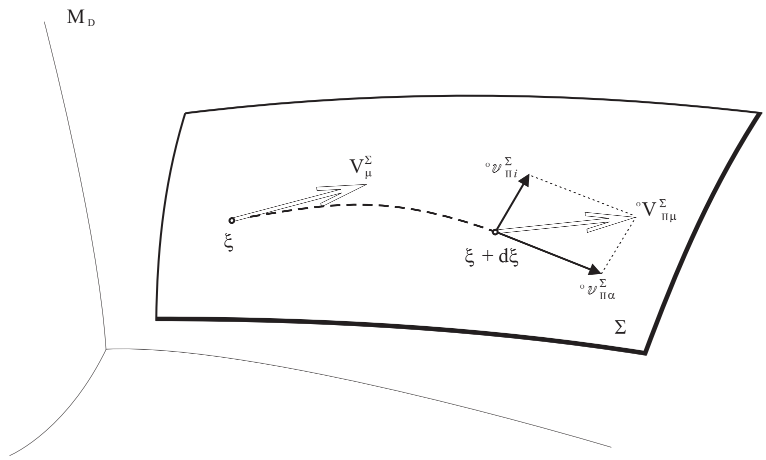

Figure 2: *Definition of the induced connection* $^{\circ}\Gamma^{\beta}_{\alpha\gamma}$ *and SFF* $^{\circ}b_{i\alpha\beta}$ *from the parallel transport of the world-sheet tangent covector. Here* $^{\circ}v^{\Sigma}_{\|\alpha} \equiv v_\alpha + {}^{\circ}\Gamma^{\beta}_{\alpha\gamma} v_\beta \, d\xi^\gamma$ *and* $^{\circ}v^{\Sigma}_{\|i} \equiv -{}^{\circ}b_{i\alpha\beta} \, v^\alpha d\xi^\beta$.

where

$$^{\circ}\delta V^{\Sigma}_{\mu} = {}^{\circ}\Gamma^{\nu}_{\mu\rho} V^{\Sigma}_{\nu} dx^\rho = {}^{\circ}\Gamma^{\nu}_{\mu\rho} V^{\Sigma}_{\nu} \partial_\gamma x^\rho d\xi^\gamma \, . \tag{3.12}$$

In the local basis, at the point $\xi + d\xi$, its world-sheet projection has the form

$$^{\circ}v^{\Sigma}_{\|\alpha} = \partial_\alpha x^\mu (\xi + d\xi) {}^{\circ}V^{\Sigma}_{\|\mu} \, . \tag{3.13}$$

Let us introduce the world-sheet **induced connection**, $^{\circ}\Gamma^{\beta}_{\alpha\gamma}$, which defines the rule for the parallel transport of the world-sheet covector v_α, along the same world-sheet line, to the projection $^{\circ}v^{\Sigma}_{\|\alpha}$. So, we have by definition

$$^{\circ}v^{\Sigma}_{\|\alpha} = v_\alpha + {}^{\circ}\delta v_\alpha \, , \qquad {}^{\circ}\delta v_\alpha = {}^{\circ}\Gamma^{\beta}_{\alpha\gamma} v_\beta d\xi^\gamma \, . \tag{3.14}$$

It produce the expression for the induced connection

$$^{\circ}\Gamma^{\gamma}_{\alpha\beta} = G^{\gamma\delta} \partial_\delta x^\mu G_{\mu\nu} ({}^{\circ}\Gamma^{\nu}_{\rho\sigma} \partial_\alpha x^\rho \partial_\beta x^\sigma + \partial_\beta \partial_\alpha x^\nu) = G^{\gamma\delta} \partial_\delta x^\mu G_{\mu\nu} {}^{\circ}D_\beta \partial_\alpha x^\nu \, , \tag{3.15}$$

where $^{\circ}D_\alpha V^\mu = \partial_\alpha x^\nu {}^{\circ}D_\nu V^\mu$ is space-time covariant derivative along world-sheet direction.

The orthogonal projection of the covector $^{\circ}V^{\Sigma}_{\|\mu}$, defines the **second fundamental form**, $^{\circ}b_{i\alpha\beta}$, trough the equation

$$n^\mu_i (\xi + d\xi) {}^{\circ}V^{\Sigma}_{\|\mu} \equiv {}^{\circ}v^{\Sigma}_{\|i} = -{}^{\circ}b_{i\alpha\beta} v^\alpha d\xi^\beta \, , \tag{3.16}$$

or explicitly

$$^{\circ}b_{i\alpha\beta} = n^\mu_i \, G_{\mu\nu} {}^{\circ}D_\beta \partial_\alpha x^\nu = -\partial_\alpha x^\nu G_{\mu\nu} {}^{\circ}D_\beta n^\mu_i \, . \tag{3.17}$$

The SFF define the extrinsic geometry.

So, the induced connection and the SFF are the world-sheet and normal projections respectively, of the space-time covariant derivative of $\partial_\alpha x^\mu$. Consequently, we have the generalization of the Gauss-Weingarten equation

$$ {}^\circ D_\beta \partial_\alpha x^\mu = {}^\circ\Gamma^\gamma_{\alpha\beta}\partial_\gamma x^\mu + {}^\circ b^i_{\alpha\beta}\, n^\mu_i \,. \tag{3.18} $$

3.3 Relations between space-time and world-sheet features

In analogy with the general rule for connection decomposition, we can decompose the induced connection and SFF. With the help of (3.5) we have

$$ {}^\circ D_\beta \partial_\alpha x^\mu = D_\beta \partial_\alpha x^\mu + {}^\circ K^\mu{}_{\nu\rho}\partial_\alpha x^\nu \partial_\beta x^\rho \,. \tag{3.19} $$

The world-sheet projection of the last equation, produces the decomposition of the induced connection

$$ {}^\circ\Gamma_{\gamma,\alpha\beta} = \Gamma_{\gamma,\alpha\beta} + {}^\circ K_{\gamma\alpha\beta}\,, \tag{3.20} $$

in terms of induced Christoffel connection and **induced torsion**

$$ \Gamma^\gamma_{\alpha\beta} = G^{\gamma\delta}\partial_\delta x^\mu\, G_{\mu\nu}\, D_\beta \partial_\alpha x^\nu \,, \tag{3.21} $$

$$ {}^\circ T_{\alpha\beta\gamma} = {}^\circ T_{\mu\rho\sigma}\partial_\alpha x^\mu \partial_\beta x^\rho \partial_\gamma x^\sigma = {}^\circ\Gamma_{\alpha,\beta\gamma} - {}^\circ\Gamma_{\alpha,\gamma\beta}\,, \tag{3.22} $$

where ${}^\circ K_{\gamma\alpha\beta} = \frac{1}{2}{}^\circ T_{\{\beta\gamma\alpha\}}$ is the induced contortion. Note that the torsion is a tensors, so that the corresponding relation does not have non-homogeneous term.

The orthogonal projection of (3.19) produces the decomposition of the SFF

$$ {}^\circ b_{i\alpha\beta} = b_{i\alpha\beta} + {}^\circ K_{i\alpha\beta}\,, \tag{3.23} $$

where we used the notations

$$ b_{i\alpha\beta} = n^\mu_i G_{\mu\nu} D_\beta \partial_\alpha x^\nu \,, \qquad {}^\circ T_{i\alpha\beta} = {}^\circ T_{\mu\rho\sigma} n^\mu_i \partial_\alpha x^\rho \partial_\beta x^\sigma \,, \tag{3.24} $$

and similarly as before ${}^\circ K_{i\alpha\beta} = \frac{1}{2}{}^\circ T_{\{\beta i\alpha\}}$.

When the torsion is present, the SFF is not symmetric in α, β indices. Similarly as in the case of the connection, we can write

$$ {}^\circ b_{i\alpha\beta} = {}^\circ b_{i(\alpha\beta)} + \tfrac{1}{2}{}^\circ T_{i\alpha\beta}\,. \tag{3.25} $$

Starting with the definition of the space-time covariant derivatives along world-sheet direction, ${}^\circ DV^\mu = V^\mu(\xi + d\xi) - V^\mu_\parallel$, we can obtain the relations

$$ {}^\circ D_\alpha V^\mu_\Sigma = {}^\circ\nabla_\alpha v^\beta\, \partial_\beta x^\mu + n^\mu_i\, {}^\circ b^i_{\beta\alpha} v^\beta \,, \tag{3.26} $$

$$ ({}^\circ D_\alpha V_\mu)\partial_\beta x^\mu = {}^\circ\nabla_\alpha v_\beta - v^i\, {}^\circ b_{i\beta\alpha}\,, \tag{3.27} $$

which we will need later.

4 World-sheet as a minimal surface

In this section we will present geometrical interpretation of the field equations in the absence of antisymmetric tensor field $B_{\mu\nu}$.

4.1 Mean extrinsic curvature

In the torsion free case, the SFF is symmetric in the world-sheet indices and its properties are well known in the literature. Here, we will generalize it for the spaces with Minkowski signature.

As usual, a curve is parametrized by its length parameter s, so that the unit tangent vector is $t^\mu = \frac{dx^\mu}{ds}$. If the curve lies in the world-sheet, we have $t^\mu = t^\alpha \partial_\alpha x^\mu$, with $t^\alpha = \frac{d\xi^\alpha}{ds}$. Let us denote by P_i, the 2-dimensional plane spanned by the tangent vector t^μ and the unit world-sheet normal n_i^μ. Then, $\ell_i = \Sigma \bigcap P_i$ is the i-th normal section of the world-sheet Σ. The curvature of the normal section ℓ_i, as a space-time curve, is

$$ {}^\circ k_i = {}^\circ D_s t^\mu G_{\mu\nu} n_i^\nu \,, \tag{4.1} $$

where ${}^\circ D_s t^\mu \equiv \dot{x}^\nu {}^\circ D_\nu t^\mu$, is the covariant derivative along the curve. It produces the following expression

$$ {}^\circ k_i = {}^\circ b_{i\beta\alpha}\, t^\alpha t^\beta = {}^\circ b_{i(\beta\alpha)}\, t^\alpha t^\beta = \frac{{}^\circ b_{i(\beta\alpha)} d\xi^\alpha d\xi^\beta}{G_{\alpha\beta} d\xi^\alpha d\xi^\beta} \,. \tag{4.2} $$

The curvature ${}^\circ k_i$ depends on the direction of the tangent vector t^α and only on the symmetric part of the SFF.

In Euclidean spaces, the maximal and minimal values of ${}^\circ k_i$ are the principal curvatures. The corresponding directions, defined by t^α, are the principal directions. The principal curvatures are eigenvalues of the SFF and corresponding eigenvectors define the principal directions.

In the present case with Minkowski space-time, the curvature ${}^\circ k_i$, (4.2), is divergent in the light-cone directions. Consequently, the extremely values do not exist. Still, we can obtain the necessary information from the eigenvalue problem

$$ ({}^\circ b^i_{\alpha\beta} - {}^\circ \kappa^i G_{\alpha\beta}) v_i^\beta = 0 \,. \tag{4.3} $$

The eigenvalues of the quadratic forms ${}^\circ b^i_{\alpha\beta}$ with respect to the metric $G_{\alpha\beta}$, we define as a principal curvatures in Minkowski space-time. They are the solutions of the condition $\det({}^\circ b^i - {}^\circ\kappa^i G)_{\alpha\beta} = 0$, or explicitly

$$ {}^\circ\kappa^i_{0,1} = {}^\circ H^i \pm \sqrt{({}^\circ H^i)^2 - {}^\circ K^i} \,. \tag{4.4} $$

Here

$$^\circ H^i = \tfrac{1}{2} G^{\alpha\beta}\, {}^\circ b^i_{\alpha\beta} = \tfrac{1}{2}({}^\circ\kappa^i_0 + {}^\circ\kappa^i_1)\,, \tag{4.5}$$

is the trace of the SFF, known in the literature as **mean extrinsic curvature** and $^\circ K^i = \frac{\det {}^\circ b^i_{\alpha\beta}}{\det G_{\alpha\beta}} = {}^\circ\kappa^i_0\, {}^\circ\kappa^i_1$ (no summation over i) is Gauss curvature.

The surface defined by the equation $^\circ H^i = 0$ is **minimal surface**. The name stems from the fact that in the Riemann space-time the equation $H_i = 0$ define the surface of minimal area $P_2 = \int d^2\xi \sqrt{-\det G_{\alpha\beta}}$, for the fixed boundary. In fact, for this surface the first variation of the area vanishes

$$\delta P_2 = -2 \int d^2\xi \sqrt{-\det G_{\alpha\beta}}\, H^i n^\mu_i G_{\mu\nu}\, \delta x^\nu = 0\,. \tag{4.6}$$

4.2 Local properties of the world-sheet embedded in Riemann space-time

In the absence of antisymmetric field, $B_{\mu\nu} = 0$, the equations of motion for the action (2.1), survive in the simpler form

$$[x^\mu] \equiv \nabla_- \partial_+ x^\mu + \Gamma^\mu_{\rho\sigma} \partial_+ x^\rho \partial_- x^\sigma = 0\,, \qquad [h^\pm] \equiv G_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu = 0\,, \tag{4.7}$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel connection. In this case the string does not see the torsion and it feels the target space as *Riemann space-time* of general relativity [9, 10].

The world-sheet and orthogonal projections of the equation $[x^\mu]$, obtain the forms

$$g^{\alpha\beta}(\omega^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}) = 0\,, \qquad g^{\alpha\beta} b^i_{\beta\alpha} = 0\,. \tag{4.8}$$

Here, $\Gamma^\gamma_{\alpha\beta}$ is induced world-sheet Christoffel connection (3.21) and $b^i_{\alpha\beta}$ is corresponding SFF (3.24). By $\omega^\gamma_{\alpha\beta}$ we denote intrinsic world-sheet connection.

The $[h^\pm]$ equation can be written in the form

$$G_{\pm\pm} \equiv e^\alpha_\pm e^\beta_\pm G_{\alpha\beta} = 0\,, \tag{4.9}$$

where $G_{\alpha\beta} = G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$ is world-sheet induced metric. Because in the light-cone frame, the intrinsic metric is off-diagonal, $g_{\pm\pm} = 0$, we have $G_{ab} = \lambda g_{ab}$, $(a, b \in \{+,-\})$, or equivalently

$$G_{\alpha\beta} = \lambda g_{\alpha\beta}\,. \tag{4.10}$$

Multiplying the last equation with $g^{\alpha\beta}$, we obtain the expression $\lambda = \frac{1}{2} g^{\alpha\beta} G_{\alpha\beta}$, so that $\sqrt{-G} = \sqrt{-g}\frac{1}{2} g^{\alpha\beta} G_{\alpha\beta}$. The last relation connects the Polyakov and Nambu-Goto expressions for the string action. We can rewrite (4.10) in the form

$$\frac{G_{\alpha\beta}}{\sqrt{-G}} = \frac{g_{\alpha\beta}}{\sqrt{-g}}\,, \tag{4.11}$$

so that, because of the conformal invariance, only the metric densities are related.

From (4.10) follows a relation between intrinsic connection $\omega^\gamma_{\alpha\beta}$ and induced one $\Gamma^\gamma_{\alpha\beta}$ which is also a solution of the first equation (4.8). Therefore, both intrinsic metric tensor and connection, are equal to the induced ones from the space-time, up to the conformal factor λ. This is expected result, because of the conformal invariance of the action.

With the help of (4.10) the second equation (4.8) becomes

$$H^i \equiv \tfrac{1}{2} G^{\alpha\beta} b^i_{\beta\alpha} = 0\,. \tag{4.12}$$

Therefore, the vanishing of all MECs, are local properties of minimal world-sheet in Riemann space-time.

Out of the initial, D components of $[x^\mu]$ equations, two define intrinsic connection in terms of the induced one and $D-2$ turn all MECs to zero.

5 World-sheet as a CT-dual surface

Let us stress that above considerations are torsion independent, because the antisymmetric part of the SFF disappears from (4.2) and (4.5). In this section we are going to include the torsion contribution and generalize above results.

5.1 CT-duality between mean extrinsic curvature and extrinsic mean torsion

Let us first generalize the eigenvalue problem, and then offer its geometrical interpretation. We introduce the **dual eigenvalue** problem, such that linear transformation of the vector v^α with matrix ${}^\circ b^i_{\alpha\beta}$ is proportional to the two dimensional dual vector ${}^* v_\alpha = \sqrt{-G_2}\,\varepsilon_{\alpha\beta}\, v^\beta$ ($G_2 = \det G_{\alpha\beta}$)

$${}^\circ b^i_{\alpha\beta}\, v^\beta = {}^*\kappa^i\, {}^* v_\alpha\,, \tag{5.1}$$

or equivalently

$$\left({}^\circ b^i_{\alpha\beta} - {}^*\kappa^i\, \varepsilon_{\alpha\beta}\sqrt{-G_2}\right) v^\beta_i = 0\,. \tag{5.2}$$

It is similar to (4.3), but for the completeness we also need the eigenvalues of the quadratic forms ${}^\circ b^i_{\alpha\beta}$ with respect to the antisymmetric tensor $\varepsilon_{\alpha\beta}\sqrt{-G_2}$.

We can formulate the dual eigenvalue problem (5.2), as an ordinary eigenvalue problem $({}^* b^i_{\alpha\beta} - {}^*\kappa^i G_{\alpha\beta}) v^\alpha_i = 0$, if we introduce the dual SFF

$${}^* b^i_{\alpha\beta} = \frac{G_{\alpha\gamma}\varepsilon^{\gamma\delta}}{\sqrt{-G_2}}\, {}^\circ b^i_{\delta\beta}\,. \tag{5.3}$$

The solutions of the condition $\det({}^* b^i - {}^*\kappa^i\, G)_{\alpha\beta} = 0$, have the form

$${}^*\kappa^i_{0,1} = {}^* H^i \pm \sqrt{({}^* H^i)^2 + {}^\circ K^i}\,. \tag{5.4}$$

In analogy with the previous case, we will call them dual principal curvatures, and the variable

$$^*H^i = \tfrac{1}{2}(^*\kappa_0^i + {}^*\kappa_1^i) = \tfrac{1}{2}G^{\alpha\beta *}b^i_{\alpha\beta} = \tfrac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}{}^\circ b^i_{\beta\alpha} = \tfrac{1}{4}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}{}^\circ T^i_{\beta\alpha}\,, \tag{5.5}$$

the **dual mean extrinsic curvature**. Here $^\circ K^i = \frac{\det {}^\circ b^i_{\alpha\beta}}{\det G_{\alpha\beta}} = {}^*\kappa_0^i {}^*\kappa_1^i$ (no summation over i) is the same Gauss curvature as before.

Let us turn to the geometrical meaning of the DMEC. In the case when t_1^μ and t_2^μ are world-sheet tangent vectors (Fif. 1) (note that all geodesics ℓ_1, ℓ_2, $\tilde\ell_1$ and $\tilde\ell_2$ still could be space-time curves) we can rewrite (3.4) in the form

$$^\circ T^\mu \equiv \frac{x^\mu(D_2) - x^\mu(D_1)}{2dP_{12}} = \frac{\varepsilon^{\beta\alpha}}{4\sqrt{-G_2}}\,{}^\circ T^\mu_{\rho\sigma}\partial_\alpha x^\rho \partial_\beta x^\sigma\,. \tag{5.6}$$

Here, $dP_{12} = \sqrt{-G_2}\det(\frac{\partial\xi^\alpha}{\partial s_r})d\ell_1 d\ell_2$ is area of the parallelogram, spanned by the vectors $\ell_1^\mu = t_1^\mu d\ell_1$ and $\ell_2^\mu = t_2^\mu d\ell_2$. We can conclude that $^\circ T^\mu$ does not depend on the directions t_1^μ and t_2^μ, and on the lengths $d\ell_1$ and $d\ell_2$. So, we will refer to this variable as the **mean torsion**. Its world-sheet projection is induced mean torsion

$$^\circ T_\gamma = {}^\circ T^\mu G_{\mu\nu}\partial_\gamma x^\nu = \frac{\varepsilon^{\beta\alpha}}{4\sqrt{-G_2}}\,{}^\circ T_{\gamma\alpha\beta}\,. \tag{5.7}$$

Its normal projection is the **extrinsic mean torsion** (EMT)

$$^\circ T_i = {}^\circ T^\mu G_{\mu\nu} n_i^\mu = \frac{\varepsilon^{\beta\alpha}}{4\sqrt{-G_2}}\,{}^\circ T_{i\alpha\beta} = {}^*H_i\,, \tag{5.8}$$

which is exactly the same variable as DMEC, defined above in (5.5).

We define the **CT-duality** (**C**urvature-**T**orsion duality), which maps SFF to dual SFF, $^\circ b^i_{\alpha\beta} \to {}^*b^i_{\alpha\beta}$, and interchanges the role played by the symmetric and the antisymmetric parts of the SFF. Consequently, under CT-duality MEC maps to DMEC, allowing the exchange of the mean extrinsic curvature and extrinsic mean torsion.

The self-dual and self-antidual configurations

$$^\circ H^i = \pm {}^*H^i\,, \qquad \Leftrightarrow \qquad {}^\circ H^i = \pm {}^\circ T^i\,, \tag{5.9}$$

correspond to the following conditions on the SFF

$$(G^{\alpha\beta} \mp \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}}){}^\circ b^i_{\beta\alpha} = 0\,. \tag{5.10}$$

The equations (5.9) and (5.10) define **CT-dual (antidual) surfaces**. In the torsion free case, they turn to the standard minimal surface condition, $^\circ H^i = 0$.

5.2 Local properties of the world-sheet embedded in Riemann-Cartan space-time

Let us applay the results of the previous subsection to the action (2.1) when both metric tensor $G_{\mu\nu}$ and the antisymmetric field $B_{\mu\nu}$ are present, so that we have complete equations of motion (2.5) and (2.6). In this case, the string feels the target space as *Riemann-Cartan space-time.*

As well as in the case of Riemann space-time, the same relation between metric tensors, (4.10), follows from the $[h^{\pm}]$ equation. We can rewrite the $[x^{\mu}]$ equation in the form

$$g^{\alpha\beta}(D_{\beta}\partial_{\alpha}x^{\mu} - \omega^{\gamma}_{\alpha\beta}\partial_{\gamma}x^{\mu}) = \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B^{\mu}_{\alpha\beta}\,. \tag{5.11}$$

Its world-sheet projection produces

$$g^{\alpha\beta}(\omega^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta}) = -\frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B^{\gamma}_{\alpha\beta}\,, \tag{5.12}$$

where $B^{\gamma}_{\alpha\beta}$ is the world-sheet torsion, induced from the target space. Because it is totally antisymmetric, in two dimensions it vanishes, $B^{\gamma}_{\alpha\beta} = 0$. So, we obtain the same first equation (4.8), as in the case of Riemann space-time. Again, both two dimensional intrinsic metric tensor and two dimensional intrinsic connection, up to the conformal factor, are induced from the target space. Note that the world-sheet is torsion free while the space-time is not.

The orthogonal projection of the $[x^{\mu}]$ equation takes the form

$$g^{\alpha\beta}b_{i\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B_{i\alpha\beta}\,, \tag{5.13}$$

or equivalently

$$(g^{\alpha\beta} - \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}})b_{-i\beta\alpha} = 0\,. \tag{5.14}$$

With the help of (4.10), we can rewrite it in terms of induced metric

$$(G^{\alpha\beta} - \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G_2}})b_{-i\beta\alpha} = 0\,, \qquad \Leftrightarrow \qquad H_i = {}^*H_{-i}\,. \tag{5.15}$$

Therefore, according to the local property, the world-sheet embedded in Riemann-Cartan space-time is *CT-dual surface* and the field equations have a form of CT-duality.

As a consequence of the relation $\Gamma^{\mu}_{+\rho\sigma} = \Gamma^{\mu}_{-\sigma\rho}$, we have $b_{+i\alpha\beta} = b_{-i\beta\alpha}$, so that with respect to $\Gamma^{\mu}_{+\rho\sigma}$ the world-sheet is CT-antidual surface.

6 Conclusions

In this paper, we investigated the bosonic string propagating in the nontrivial background. We described the classical field equations of in term of world-sheet local properties valid in the neighborhood of every given point.

We started with geometry of the surface embedded into Riemann-Cartan space-time. We clarified the meaning of MEC in Minkowski space-time ${}^\circ H^i$ (see (4.5)), and introduced the concept of DMEC ${}^* H^i$ (5.5) as orthogonal projection of the mean torsion (see (5.7) and (5.8)). We defined CT-duality which maps MEC to DMEC. The presence of torsion generalize the equation of minimal surfaces. Instead of the standard equation ${}^\circ H_i = 0$, we introduced CT-dual (antidual) surface defined by the self-duality (self-antiduality) conditions, ${}^\circ H_i = \pm {}^* H_i$.

Then we considered the equations of motion (2.5)-(2.6). As a consequence of the second equation, the intrinsic metric tensor is equal to the induced one up to the conformal factor λ, because the theory is conformally invariant.

The first equation, which has been obtained by variation with respect to x^μ, have D components. Two of them determine the contracted intrinsic connection in terms of the corresponding induced one. They are not independent, because they follow from the $[h^\pm]$ equation.

The other $D-2$ are of the form $H_i = {}^* H_{-i}$, where H_i is MEC and ${}^* H_{-i}$ is DMEC. They define world-sheet as CT-dual surface of the Riemann-Cartan space-time. In the particular case, —the vanishing torsion — the field equations turn to the equations of minimal world-sheet, $H_i = 0$, of the Riemann space-time.

The dilaton field is origin of nonmetricity [3]. It broke the conformal invariance and produce additional field equation with respect to conformal part of world-sheet metric. The analysis of the dilaton contribution to the local properties of the world-sheet, which is technically more complicated, has been investigated in [3].

A World-sheet geometry

It is useful to parameterize the intrinsic world-sheet metric tensor $g_{\alpha\beta}$, with the light-cone variables (h^+, h^-, F) (see the papers [8, 11, 12])

$$g_{\alpha\beta} = e^{2F} \hat{g}_{\alpha\beta} = \tfrac{1}{2} e^{2F} \begin{pmatrix} -2h^- h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix} . \tag{A.1}$$

The world-sheet interval

$$ds^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = 2 d\xi^+ d\xi^- , \tag{A.2}$$

can be expressed in terms of the variables

$$d\xi^{\pm} = \frac{\pm 1}{\sqrt{2}} e^F (d\xi^1 - h^{\pm} d\xi^0) = e^F d\hat{\xi}^{\pm} = e^{\pm}{}_{\alpha} d\xi^{\alpha} \,. \tag{A.3}$$

The quantities $e^{\pm}{}_{\alpha}$ define the light-cone one form basis, $\theta^{\pm} = e^{\pm}{}_{\alpha} d\xi^{\alpha}$, and its inverse define the tangent vector basis, $e_{\pm} = e_{\pm}{}^{\alpha}\partial_{\alpha} = \partial_{\pm}$. We will use the relations

$$\eta^{ab} e_a{}^{\alpha} e_b{}^{\beta} = e_+{}^{\alpha} e_-{}^{\beta} + e_-{}^{\alpha} e_+{}^{\beta} = g^{\alpha\beta} \,, \qquad \varepsilon^{ab} e_a{}^{\alpha} e_b{}^{\beta} = e_+{}^{\alpha} e_-{}^{\beta} - e_-{}^{\alpha} e_+{}^{\beta} = \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} \,, \tag{A.4}$$

where $a, b \in \{+, -\}$.

In the tangent basis notation, the components of the arbitrary vector V_{α} have the form

$$V_{\pm} = e^{-F} \hat{V}_{\pm} = e_{\pm}{}^{\alpha} V_{\alpha} = \frac{\sqrt{2} e^{-F}}{h^- - h^+} (V_0 + h^{\mp} V_1) \,. \tag{A.5}$$

The world-sheet covariant derivatives on tensor X_n are

$$\nabla_{\pm} X_n = (\partial_{\pm} + n\omega_{\pm}) X_n \,, \tag{A.6}$$

where the number n is sum of the indices, counting index $+$ with 1 and index $-$ with -1, and

$$\omega_{\pm} = e^{-F} (\hat{\omega}_{\pm} \mp \hat{\partial}_{\pm} F) \,, \qquad \hat{\omega}_{\pm} = \mp \frac{\sqrt{2}}{h^- - h^+} h^{\mp\prime} \,, \tag{A.7}$$

are two dimensional Riemannian connections.

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